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ON SUBSETS WITH SMALL PRODUCT IN TORSION-FREE GROUPS

Y. O. HAMIDOUNE*, A. S. LLADÓ† and O. SERRA†

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Let G be a nonabelian torsion-free group. Let C be a finite generating subset of G such that $1 \in C$. We prove that, for all subsets B of G with $|B| \ge 4$, we have $|BC| \ge |B| + |C| + 1$.

In particular, a finite subset X with cardinality $k \ge 4$ satisfies the inequality $|X^2| \le 2|X|$ if and only if there are elements $x, r \in G$, such that the following two conditions hold:

(i) $xr = rx$.

(ii) $Xx = \{1, r, \cdots, r^k\} \setminus \{c\}$ where $c \in \{1, r\}$.

1. Introduction

Throughout the paper G denotes a torsion-free group written multiplicatively. A subset of the form $\{ar^i | 1 \le i \le m\}$ for some $a, r \in G$ and $m \in \mathbb{N}$ is said to be a *left progression* of ratio r or a left r-progression. Similarly, a set of the form $\{r^i b | 1 \le i \le m\}$ for some $b \in G$ is a *right r-progression.* A set which is both a left and a right r-progression is simply a *progression.* Notice that a left progression containing 1 is a progression.

Given two finite sets $B, C \subset G$, we write $BC = \{bc | b \in B, c \in C\}$. One of the basic problems in Additive Theory consists in giving lower bounds for $|BC|$ in terms of the cardinalities of the two sets B and C. The *inverse problem* consists in deriving structural properties of the two sets from the knowledge of a bound for the cardinality of their product. There are several important results of this kind for torsion free groups.

When G is cyclic, the $(3k-4)$-Theorem of Freiman [2, 9] states that

$$|B^2| \ge 3|B| - 3,$$

unless B is contained in a progression of size at most $2|B| - 2$. The validity of this result for abelian torsion-free groups follows by results in [2].

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The $(3k$–$4)$-Theorem was generalized to the product of distinct sets by Freiman [11] and by Lev and Smelianski [7]. For abelian torsion-free groups with dimension greater than 1, good lower bounds for $|BC|$ are due to Ruzsa [10].

Non abelian torsion-free groups are good candidates for the validity of the $(3k-4)$-Theorem, but this question is still open. Only few less precise results are known in the non abelian case. By a result of Kemperman [6],

$$|BC| \geq |B| + |C| - 1.$$

In [1] Brailovsky and Freiman proved that, for $|B|,|C|\geq 2$,

$$|BC| \geq |B| + |C|$$

unless there are $b \in B^{-1}$ and $c \in C^{-1}$ such that both bB and Cc are progressions with the same ratio.

Using a different approach, one of the authors obtained in [4] a common generalization of the last result and Vosper's Theorem [12, 8].

Our main result is the following:

Let G be a nonabelian torsion-free group. Let C be a generating subset of G such that $1\in C$. Then, for all subsets B such that $|B|\geq 4$,

$$|BC| \geq |B| + |C| + 1.$$

In particular, a finite subset X with cardinality $k \geq 4$ satisfies the inequality $|X^2| \leq 2|X|$ if and only if there are elements $x, r \in G$, such that the following two conditions hold:

(i) $xr = rx$.

(ii) $Xx = \{1, r, \cdots, r^k\} \setminus \{c\}$ where $c \in \{1, r\}$.

2. Preliminaries

Let G be an infinite group and let $1 \in C$ be a finite generating subset of G. For $X \subset G$, we shall write

$$\partial X = XC \setminus X.$$

We write $\partial_C X$ if the reference to C is not clear from the context. The *k-isoperimetric number* of C is

$$\kappa_k(C) = min\{|\partial X| : X \subset G, k \leq |X| < \infty\}.$$

A finite subset X of G is a *k-critical set* of C if $|X| \geq k$ and $|\partial X| = \kappa_k(C)$. A *$k$-atom* of C is a k-critical set of C with minimal cardinality. We denote by $\alpha_k(C)$ the cardinality of a k-atom of C.

The following lemma is a special case of a result proved in [4]. We include here a short proof for the benefit of the reader.

Lemma 1. ([4]) *Let $1 \in C$ be a finite generating subset of a torsion-free group G. Let F be a k-critical set and A a k-atom of $C \subset G$.*

Then either $A \subset F$ or $|A \cap F| \le k-1$.

In particular, for each $x \in G \setminus \{1\}$, we have $|A \cap xA| \le k-1$.

Proof. We write $\varepsilon X = G \setminus (X \cup \partial X)$. Suppose that $|A \cap F| \ge k$ and $A \not\subset F$. Since A is a k-atom, we have $|\partial(A \cap F)| > |\partial A|$. Therefore,

$$|\partial A \cap F| + |\partial A \cap \partial F| + |\partial A \cap \varepsilon F| = |\partial A| < |\partial(A \cap F)| \le |A \cap \partial F| + |\partial A \cap F| + |\partial A \cap \partial F|,$$

which imply

$$|\partial A \cap \varepsilon F| < |A \cap \partial F|. \tag{1}$$

On the other hand, since F is a k-critical set, we have $|\partial F| \le |\partial(A \cup F)|$, which leads to

$$|\partial F \cap A| + |\partial F \cap \partial A| + |\partial F \cap \varepsilon A| = |\partial F| \le |\partial(F \cup A)| \le |\varepsilon F \cap \partial A| + |\partial F \cap \varepsilon A| + |\partial F \cap \partial A|.$$

Hence, $|\partial F \cap A| \le |\varepsilon F \cap \partial A|$ contradicting (1).

To prove the second part of the Lemma, note that xA is also a k-atom, for each $x \in G$. In particular, if $|A \cap xA| \ge k$ for some $x \in G \setminus \{1\}$, we must have $A = xA$, contradicting $|A| < \infty$. ∎

The Lemma above provides a simple proof of the Cauchy- Davenport inequality for torsion-free groups deduced in [1] from a result by Kempermann [6].

Corollary 2. ([1]) *Let G be a torsion-free group and let B, C be finite nonempty subsets of G. Then $|BC| \ge |B| + |C| - 1$.*

Proof. By exchanging C by Cc^{-1} for some $c \in C$ if necessary, we may assume that $1 \in C$. Let A be a 1-atom of C such that $1 \in A$. If there is an element $x \in A \setminus \{1\}$, we have $|xA \cap A| \ge 1$, contradicting Lemma 1. Therefore $A = \{1\}$. Now $|BC| - |B| \ge \kappa_1(C) = |AC| - |A| = |C| - 1$. ∎

The following easy lemma will be needed.

Lemma 3. *Let $1 \in C$ be a finite generating subset of a torsion-free group G. Let B be a finite subset generating a proper subgroup of G and $|B| \ge 3$. Then*

$$|BC| \ge |B| + |C| + 1.$$

Proof. Partition $C = C_1 \cup C_2 \cup \cdots \cup C_j$, where each C_i is the nonempty intersection of C with some right coset of the subgroup generated by B. Necessarily $j \ge 2$.

By Corollary 2, $|BC| \ge \sum_{i=1}^{j} (|C_i| + (|B| - 1)) = |C| + j(|B| - 1) \ge |B| + |C| + 1.$ ∎

The following property of k-atoms will be often used.

Lemma 4. *Let $1 \in C$ be a finite generating subset of a torsion-free group G. Let A be a k-atom of C such that $|A| \geq k+1$. Then*

$$|zC^{-1} \cap A| \geq 2 \quad \forall z \in AC. \tag{2}$$

Moreover

$$|A|(|C|-2) \geq 2\kappa_k(C) \tag{3}$$

Proof. Note that, as $1 \in C$, we have $|zC^{-1} \cap A| \geq 1$ for all $z \in AC = A \cup \partial A$.

Suppose $|zC^{-1} \cap A| = 1$ for some $z \in A$. Then $\partial(A \setminus \{z\}) \subset \partial A$ contradicting the minimality of A. On the other hand, if $zC^{-1} \cap A = \{u\}$ for some $z \in \partial A$, then $\partial(A \setminus \{u\}) \subset (\partial A \setminus \{z\}) \cup \{u\}$, contradicting again the definition of a k-atom. This proves (2).

Set $\mu(A) = \sum_{a \in A} |aC \cap A|$. We clearly have

$$|A||C| = \mu(A) + \sum_{a \in A} |aC \setminus A| = \mu(A) + \sum_{z \in \partial(A)} |zC^{-1} \cap A|.$$

It follows by (2) that

$$|A||C| \geq \mu(A) + 2\kappa_k(C).$$

Now (3) follows since, again by (2), $\mu(A) = \sum_{a \in A} |aC^{-1} \cap A| \geq 2|A|$. ∎

We need the following lemma.

Lemma 5. ([4]) *Let C be a finite generating subset of a torsion-free group G such that $|C| \geq 3$ and $1 \in C$. Let A be a 2-atom of C. Then $|A| \leq |C| - 1$.*

Proof. We may assume that $1 \in A$. If $|A| = 2$ there is nothing to prove. Suppose $|A| > 2$.

By Lemma 4, for each $x \in A$, we must have $xC^{-1} \cap A \neq \{x\}$. Therefore we can define a map $f : A \to C \setminus \{1\}$ such that $x(f(x))^{-1} \in A$ for each $x \in A$.

Let us show that such a map is injective. Indeed, $f(x) = f(y) = c$ and $x \neq y$ would imply $\{x, y\} \subseteq Ac^{-1} \cap A$ and therefore $\{x, y, xc, yc\} \subset A$. Then, for $r = xy^{-1}$, we would have $\{x, xc\} \subset rA \cap A$, contradicting Lemma 1. ∎

The following consequence of Lemma 5 is basically equivalent to the result of Brailovski and Freiman mentioned in the introduction, cf. [1, 4].

Corollary 6. *Let G be a torsion-free group generated by a finite subset C containing 1 which is not a progression. Then $\kappa_2(C) \geq |C|$.*

Proof. Suppose the contrary and choose a counter-example with minimal $|C|$.

Let A be a 2-atom of C containing 1.

Suppose first that $|A|\geq 3$. By Lemma 3, A generates G.

Since $|C^{-1}A^{-1}|=|AC|\leq|C|+|A|-1$, we have $\kappa_2(A^{-1})\leq|A|-1$. Note that A can not be a progression, since otherwise there is $r\in A\setminus\{1\}$ such that $|rA\cap A|\geq 2$, contradicting Lemma 1. By Lemma 5, $|A|\leq|C|-1$, contradicting the minimality of $|C|$.

Hence, $|A|=2$. Set $A=\{1,r\}$. Partition $C=C_1\cup C_2\cup\cdots\cup C_j$, where each C_i is a maximal right r-progression. The maximality implies that $\{1,r\}C_i\cap\{1,r\}C_j=\emptyset$, for $i\neq j$. Now we have $\kappa_2(C)=|AC|-2=(|C_1|+1)+\cdots+(|C_j|+1)-2=|C|+j-2\geq|C|$, a contradiction. ∎

3. The case $|C|=3$

We first consider the situation for generating sets of cardinality 3.

Lemma 7. *Let G be a nonabelian torsion-free group and let $C=\{1,x,y\}$ be a generating set of G such that $\kappa_3(C)\leq|C|$. Let A be a 3-atom of C.*

Then $|A|=3$.

Moreover exactly one of the following conditions holds

(i) $y=xyx$;

(ii) $x=yxy$;

(iii) $x^2=y^2$ *and there is* $a\in G$ *such that* $A=aC$.

Proof. First note that, in a torsion-free group, at most one of the relations can hold. Indeed, $y=xyx$ and $x^2=y^2$ imply $y^2=xyxxyx=y^6$. On the other hand, if $y=xyx$ and $x=yxy$, then $y=yxy^2x$ and therefore $x^{-2}=y^2$. Hence, $x^{-2}=y^2=xyx^2yx=x^2$.

We shall now prove that one of the relations is satisfied.

Since C generates a nonabelian group, C is not a progression. By Corollary 6, $\kappa_2(C)\geq|C|$.

Let us show that $\alpha_3(C)=3$.

Let A be a 3-atom of C containing 1.

Suppose that $|A|\geq 4$. The (3) implies $|A|\geq 6$. By (2), for every $a\in A$, there are $s_a,t_a\in C^{-1}\setminus\{1\}$ such that $A_a=\{a,as_a,as_at_a\}\subset A$. Since C is not a progression, we have $s_at_a\neq 1$. Therefore $|A_a|=3$. Since $|C\setminus\{1\}|=2$, there are four choices for the ordered pair (s_a,t_a). Since $|A|>4$, there are two distinct elements $a,b\in A$ such that $s_a=s_b$ and $t_a=t_b$. Now $(ba^{-1})A\cap A\supset A_b$, contradicting Lemma 1.

Hence, $|A|=3$.

Let us show that $|A\cap Az|\leq 1$ for each $z\in G\setminus\{1\}$. By Corollary 6, if $|\{1,z\}A^{-1}|\leq|A|+1$ then A is a z-progression. It follows that A generates a proper (cyclic) subgroup of G. By Lemma 3, $|AC|\geq|A|+|C|+1$, a contradiction.

In particular, $|A\cap Ax|\le 1$ and $|A\cap Ay|\le 1$ and $|Ax\cap Ay|\le 1$. Now,

$$6=|AC|=9-|A\cap Ax|-|A\cap Ay|-|Ax\cap Ay|+|A\cap Ax\cap Ay|.$$

It follows that $|A\cap Ax\cap Ay|=0$ and $|A\cap Ax|=|A\cap Ay|=|Ax\cap Ay|=1$. Therefore we have one of the following cases:

(i) $A=u\{1,x,xy\}$. Clearly $\{1,x,xy\}$ is a 3-atom. Now $\{x,x^2,xyx\}\cap\{y,xy,xy^2\}\neq\emptyset$. Necessarily $xyx=y$.

(ii) $A=u\{1,y,yx\}$. We obtain similarly $yxy=x$.

(iii) $A=u\{1,x,y\}$. Clearly $\{1,x,y\}$ is a 3- atom. Now $\{x,x^2,yx\}\cap\{y,xy,y^2\}\neq\emptyset$. Hence we must have $x^2=y^2$.

This completes the proof. ∎

Lemma 8. *Let G be a nonabelian torsion-free group and let $C=\{1,x,y\}$ be a generating set of G. Then $\kappa_4(C)\ge|C|+1$,*

Proof. Assume on the contrary that $\kappa_4(C)\le|C|$.

Since $\kappa_3(C)\le\kappa_4(C)$, we may assume by Lemma 7 that either $y=xyx$ or $x=yxy$ or $x^2=y^2$.

If $x=yxy$ (resp. $x=yxy$), then $(yx^{-1})^2=(x^{-1})^2$ (resp. $(xy^{-1})^2=(y^{-1})^2$). Since $\kappa_4(C)=\kappa_4(Cx^{-1})=\kappa_4(Cy^{-1})$, we may assume that $x^2=y^2$.

Let A be a 4-atom of C.

Let us first prove that there is $a\in A$ such that $aC^{-1}\cap A=\{a\}$. Assuming the contrary, we can form a sequence $\{a_i,i\in\mathbb{N}\}$ of elements in A such that $a_{i+1}(a_i)^{-1}\in C\setminus\{1\}$ for each $i\ge 1$. Since A is finite, there are indices j and $m\ge 2$ such that $a_{j+m}=a_j$. Hence, there is a sequence $c_1,\dots,c_m$ of elements in $\{x,y\}$ such that $c_1\cdots c_m=1$. Since both x^2 and y^2 belong to the center of the group, the above relation implies a relation of the form $x^sy(xy)^r=1$, where $s,r\ge 0$. It follows that $y(xy)^rx^s=1$. By multiplying these two relations, we get $1=x^sy(xy)^ry(xy)^rx^s=x^{2(s+2r+1)}$, contradicting that G is torsion free.

Hence, there is $a\in A$ such that $aC^{-1}\cap A=\{a\}$. Let $A'=A\setminus\{a\}$. By the choice of a, we have $\partial A'\subset\partial A$. Since A is a 4-atom, we must have $|A'|=3$. As $\kappa_3(C)=\kappa_4(C)$, A' is a 3-atom of C. By Lemma 7, we may assume $A'=C$ and $A=\{1,x,y,a\}$.

Suppose that $1\in aC$. We may assume $a=x^{-1}$. Thus, $x^{-1}y\in\partial(A)=\{xy,yx,y^2\}$. This forces $x^{-1}y=yx$ which has been shown to be a relation incompatible with $x^2=y^2$.

If $1\notin aC$ then $C^{-1}\cap A=\{1\}$. As argued before, $A\setminus\{1\}$ is a 3-atom of C. By Lemma 7, $A\setminus\{1\}=vC$ for some $v\in G$. Thus, $v\{1,x,y\}=\{x,y,a\}$ and therefore $v\in\{x,y,a\}$. The only possibility is $v=a$. This forces $a\{x,y\}=\{x,y\}$, a contradiction. The proof is complete. ∎

4. The main result

Let us start with the following lemma.

Lemma 9. *Let C be a generating subset of a nonabelian torsion free group G such that $1\in C$ and $|C|\geq 4$. If $\kappa_2(C)\leq |C|$, then*

$$\alpha_2(C) = 2. \tag{4}$$

Moreover, C is the union of two right progressions.

Proof. Suppose the result false and choose a counterexample with minimal $|C|$.

Let A be a 2-atom of C containing 1.

By Lemma 5, $|A| \leq |C|-1$. By Lemma 3, A generates the whole group G. The inequality $|C^{-1}A^{-1}|=|AC|\leq |A|+|C|$ implies that $\kappa_2(A^{-1})\leq \kappa_4(A^{-1})\leq |A|$. Hence, by the minimality of $|C|$, the 2-atoms of A^{-1} have cardinality 2.

Let $A'=\{1,r\}$ be a 2-atom of A^{-1}. From $|A'A^{-1}|\leq 2+|A|$ it follows that $A^{-1}=\{1,r^{-1},\cdots,r^{-k}\}\cup\{y^{-1},r^{-1}y^{-1},\cdots,r^{-m}y^{-1}\}$ for some nonnegative integers k,m with $k+m+2=|A|$ and $y,r\in G$. We may assume that $k\geq m$.

By Lemma 8, we have $|A|\geq 4$. Observe that $|A\cap rA|\geq k$ and $|A\cap y^{-1}A|\geq m+1$. Hence, either $|A\cap rA|\geq 2$ or $|A\cap y^{-1}A|\geq 2$, contradicting Lemma 1.

Hence, $|A|=2$.

Now, if $A=\{1,r\}$, then $|AC|\leq 2+|C|$ implies that C is the union of at most two right r-progressions. Since G is nonabelian, C can not be a progression. ∎

Theorem 10. *Let C be a finite generating subset of a nonabelian torsion-free group G such that $1\in C$ and $|C|\geq 4$. Then*

$$\kappa_3(C) \geq |C| + 1.$$

Proof. Suppose the contrary and choose a counterexample with minimal $|C|$.

Let A be a 3-atom of C containing 1.

We will show that our assumption implies very tight conditions on the structure of both sets, A and C, from which we can easily derive a contradiction.

We have

$$|AC| \leq |A| + |C| \tag{5}$$

By Lemma 3, A generates G. Since $|C^{-1}A^{-1}| = |AC| \leq |A| + |C|$, we have $\kappa_4(A^{-1}) \leq |A|$. By Lemma 8, we have $|A| \geq 4$. It follows that A^{-1} is also a counterexample. By the choice of C, we have $|C|\leq |A|$.

On the other hand, we have $\kappa_2(A^{-1})\leq \kappa_4(A^{-1})\leq |A|$. By Lemma 9, A is the union of two left v-progressions for some $v\in G$, say $\{1,v,\ldots,v^k\}$ and $z\{1,v,\ldots,v^m\}$.

We must have $|A| \leq 5$, since otherwise either $|A \cap vA| \geq 3$ or $|A \cap z^{-1}A| \geq 3$, contradicting Lemma 1. Hence,

$$4 \leq |C| \leq |A| \leq 5. \tag{6}$$

By Lemma 9, C is the union of two right x-progressions for some $x \in G \setminus \{1\}$, say $C = C_1 \cup C_2$. We may assume $C_1 \subset \{1, x, x^2, x^3\}$ and $C_2 \subset \{1, x\}y$ for some $y \in G \setminus \{1\}$. We have $G = \langle x, y\rangle$ and $y \notin \langle x\rangle$.

Partition $A = A_1 \cup A_2 \cup \cdots \cup A_j$, where each A_i is the nonempty intersection of A with some left $\langle x\rangle$-coset. Without lost of generality we may assume $1 \in A_1$ and $|A_1| \geq |A_i|, i \geq 2$.

Since A generates G, we have $j \geq 2$. Let us show that $j = 2$. We have

$$|A| + |C| \geq |AC| \geq |AC_1| \geq |A| + j(|C_1| - 1).$$

Therefore, since $4 \leq |C| \leq 5$ and $|C_1| \geq |C_2|$, either $j = 2$ or $|C_1| = |C_2| = 2$. In the latter case, $C_2 = C_1 y$. The atom $A \subset AC_1$ is not a progression since it generates G. Therefore, by Corollary 6, $|AC| = |AC_1\{1, y\}| \geq |AC_1| + 2 \geq |A| + j + 2$ and we also have $j = 2$.

Let us show that both A_1 and A_2 are left x-progressions.

Suppose the contrary. We must have $|C_1| > |C_2|$, since otherwise $|AC| = |AC_1\{1, y\}| \geq |AC_1| + 2 \geq |A| + 2|C_1| + 1$, a contradiction. Therefore, $|C_1| = 3$. Using Corollary 6 again, $|AC| \geq |AC_1| = |A_1C_1| + |A_2C_1| \geq |A| + 2|C_1| - 1$, a contradiction.

We may therefore assume that $A_1 \subset \{1, x, x^2, x^3\}$ and $A_2 \subset v\{1, x\}$ with $G = \langle x, v\rangle$ and $v \notin \langle x\rangle$. Note that $|A_1| \leq 3$ since otherwise $|A \cap xA| \geq 3$ contradicting Lemma 1. Also, we must have $|C_1| \leq 3$ since otherwise $|AC| \geq |AC_1| = |A| + 2|C_1| - 2 > |A| + |C|$.

By (2), for each $z \in AC$ we have $|zC^{-1} \cap A| \geq 2$.

By taking $z = 1$ we get $2 \leq |C^{-1} \cap A| = |C_1^{-1} \cap A_1| + |C_2^{-1} \cap A_2| = 1 + |C_2^{-1} \cap A_2|$, which implies

$$|C_2^{-1} \cap A_2| \geq 1. \tag{7}$$

By taking $z = y$ we get $2 \leq |yC^{-1} \cap A| = |yC_1^{-1} \cap A_2| + |yC_2^{-1} \cap A_1| = |yC_1^{-1} \cap A_2| + 1$, which implies

$$|yC_1^{-1} \cap A_2| \geq 1. \tag{8}$$

Inequality (7) implies either $y^{-1} \in A_2$ or $y^{-1}x^{-1} \in A_2$.

Suppose that $y^{-1} \in A_2 \subset AC$. Then $2 \leq |y^{-1}C^{-1} \cap A| = |y^{-1}C_1^{-1} \cap A_2| + |y^{-1}C_2^{-1} \cap A_1|$. It is not difficult to check that all the possibilities reduce to the following: either $A_2 = y^{-1}\{1, x^{-1}\}$ or $y^2 \in \{x^{-2}, x^{-3}\}$. On the other hand, (8) implies $1 \leq |yC_1^{-1} \cap A_2| \leq |y^2\{1, x^{-1}, x^{-2}\} \cap \{x^{-1}, 1, x\}|$. Therefore we must have $A_2 = y^{-1}\{x^{-1},1\}$. Moreover, in this case we must also have $y^2 = x^2 \in C_1$. Then it can be easily checked that $|(xy)^{-1}C^{-1} \cap A| = 1$, contradicting (2) (to check that $(xy)^{-2} \neq x^2$ we can proceed as follows: $(xy)^{-2}x^{-2} = 1$ implies $1 = x^{-3}(y^{-1}x^{-1}y^{-1})(y^{-1}x^{-1}y^{-1})x^{-3} = x^{-12}$.)

Suppose now that $y^{-1} \notin A_2$. By (7) we must have $y^{-1}x^{-1} \in A_2$. By arguments similar to the ones in the previous case, (8) now leads to $A_2 = y^{-1}\{x^{-1},x^{-2}\}$ and $y^2 = x^{-2}$. In particular, we have $y \in A$. But then $|y^2C^{-1} \cap A| = 1$, contradicting again (2). This completes the proof. ∎

Theorem 10 can be reformulated as follows.

Corollary 11. *Let C be a finite generating subset of a nonabelian torsion-free group G such that $1 \in C$ and $|C| \geq 4$. Then for all $B \subset G$ with $|B| \geq 3$,*

$$|BC| \geq |B| + |C| + 1.$$

∎

Theorem 10 allows us to deduce an inverse Theorem for subsets A, B with $|AB| \leq |A| + |B|$ in a torsion-free group. We shall do this only when $A = B$. Let us introduce a definition. A subset A of a group G is said to be an *almost progression* with ratio r if there is an element $x \in G$ such that $B \cup \{x\}$ is an r-progression. According to this definition, a progression is an almost progression. If B is not a progression, x is the *hole* of the almost progression.

An almost progression A such that $|A^2| = 2|A|$ is obtained from a progression by removing its second element. This is an easy exercise for infinite cyclic groups. It is proved for groups with a prime order in [5]. We give here a proof of the following slightly more general statement.

Lemma 12. *Let G be an abelian torsion-free group. Let X and Y be finite subsets of G such that $|X|,|Y| \geq 4$. Then $|XY| \leq |Y| + |X|$ holds only if X and Y are almost progressions with a common ratio r. Moreover, if none of X and Y are progressions, then they both are almost progressions with a hole in the second position with respect to the same ratio.*

Proof. The group G, which we may assume to be generated by $X \cup Y$, is linearly orderable. We may assume that $1 \in X \cap Y$.

We first note the following remark: Let A, B be finite subsets of G such that $|AB| \leq |A| + |B|$ and let $A' = A \setminus \{\max A\} \neq \emptyset$. As $(\max A)(\max B) \in AB \setminus A'B$, we have $|A'B| \leq |A'| + |B|$. Similarly, $|A''B| \leq |A''| + |B|$ for $A'' = A \setminus \{\min A\}$.

Let $x_0 < x_1 < \ldots < x_{k-1}$ be the elements of X and set $r = \min\{x_i x_{i-1}^{-1}, 1 \leq i < k\}$. We may assume that $\{1, r\} \subset X$. By the above remark, we have $|\{1,r\}Y| \leq 2 + |Y|$. Hence, Y is the union of at most two r-progressions, say $Y = Y_1 \cup Y_2$ with $\{r, r^{-1}\}Y_1 \cap Y_2 = \emptyset$.

Let $X = X_1 \cup \ldots \cup X_j$ be a partition of X into maximal r-progressions. By the choice of r we have $(\max X_i)r < \min X_{i+1}$ for $1 \le i < j$. We consider two cases.

Case 1. X is an r-progression. By Lemma 3, G is a cyclic group. If Y is also an r-progression there is nothing to prove. Otherwise, we have $|XY| = |XY_1| + |XY_2| - |XY_1 \cap XY_2| \le |X| + |Y|$, which implies $|XY_1 \cap XY_2| \ge |X| - 2$. Hence, the hole between the two progressions Y_1 and Y_2 has length at most one and Y is an almost progression. Note that this argument requires only $|X| \ge 3$.

Case 2. X is not an r-progression. We may assume that Y is not an r-progression either. Since $\{1, r\} \subset X$, there is a part X_s with $|X_s| \ge 2$.

Assume $s < j$ and let $x = \min X_{s+1}$. Set $\bar{X} = X_s \cup \{x\}$. Let Y_2 be the r-progression containing $y = \max Y$. We must have $|Y_2| = 1$ since otherwise, $|X_s Y| \le |\bar{X} Y| - |\{xy, xyr^{-1}\}| \le |X_s| + |Y| - 1$, contradicting that Y is not an r-progression. Now, Y_1 is an r-progression and $|Y_1| \ge 3$. By Case 1, X is an almost progression. Then G is a cyclic group and it is easy to check that both X and Y must be almost r-progressions with a hole in the last but one position. Equivalently, both almost progressions have a hole in the second position with respect to the ratio r^{-1}.

A similar argument works if $s = j$ by taking X^{-1} and Y^{-1}.

This completes the proof. ∎

Corollary 13. *Let G be a torsion-free group and let X be a finite subset with cardinality $k \ge 4$. Then $|X^2| = 2|X|$ if and only if there are $x, r \in G$, such that the two following conditions hold*

(i) $xr = rx$.

(ii) $Xx = \{1, r, \cdots, r^k\} \setminus \{c\}$ *where* $c \in \{1, r\}$.

Proof. The conditions are clearly sufficient. Let us prove the necessity. By Corollary 6 we must only consider the case $|X^2| = 2|X|$.

Choose $x \in X^{-1}$. By Theorem 10, Xx generates an abelian group H. We have $xX \subset H$, since otherwise there is a partition $xX = X_1 \cup X_2$, with $1 \in X_1$ and $X_2 \not\subset H$. This would imply $|X^2| = |xXXx| \ge |X_1| + k - 1 + |X_2| + k - 1 = 3k - 2 > 2k + 1$, a contradiction.

Hence, we may apply Lemma 12 to obtain (ii).

Since $1 \in Xx$, then H is the subgroup generated by r. From $xX \subset H$, we get $Xx \subset x^{-1}Hx$. Therefore, H is a subgroup of $x^{-1}Hx$. Similarly, $Xx \subset H$ implies the opposite inclusion. It follows that $H = x^{-1}Hx$. Hence, $x^{-1}rx$ generates the cyclic group H. Therefore, either $x^{-1}rx = r$ or $x^{-1}rx = r^{-1}$.

In the first case, the group generated by x and r is abelian and the result holds by Lemma 12.

Suppose that $x^{-1}rx = r^{-1}$. By Lemma 12, xX is an almost progression. If xX is an r-progression, then so is $Xx = xX^{-1}$ and we have $|X^2| = 2|X| - 1$. Otherwise, by Lemma 12, xX must be an almost progression with a hole. Then, $Xx = xX^{-1}$ is also an almost progression with a hole. If xX has a hole in the second position, then Xx has the hole in the one before the last position, thus contradicting Lemma 12. This proves (i). ∎

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Y. O. Hamidoune

CNRS, Paris, France
yha@ccr.jussieu.fr

A. S. Lladó

Universitat Politècnica de Catalunya, Barcelona, Spain
anna@mat.upc.es

O. Serra

Universitat Politècnica de Catalunya, Barcelona, Spain
oriol@mat.upc.es